Internal **relational** parametricity, without an interval

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TYPES 2024
Copenhagen
11 June 2024

Parametricity

- $(A : \mathsf{Type}) \to A \to A$
- $(N : \mathsf{Type}) \to N \to (N \to N) \to N$
- Preservation of relations: Reynolds (1983), Wadler (1989), Plotkin–Abadi (1993), Bernardy–Jansson–Paterson (2010)
- $f : (A : \mathsf{Type}) \to A \to A$ preserves predicates:

$$f^{\mathsf{P}} : (A : \mathsf{Type})(P : A \to \mathsf{Type})(a : A) \to P\,a \to P\,(f\,A\,a)$$

fix an A and an $a : A$, *then* :

$$f^{\mathsf{P}}\,A\,(\lambda x.x = a)\,a\,\mathsf{refl} : f\,A\,a = a$$

Internal parametricity

- Pioneered by Bernardy–Moulin (2012)
- New syntax:

$$\frac{A : \mathrm{Type}}{A^{\mathsf{P}} : A \to A \to \mathrm{Type}} \qquad \frac{a : A}{a^{\mathsf{P}} : A^{\mathsf{P}}\, a\, a}$$

$$(A \to B)^{\mathsf{P}}\, f_0\, f_1 = (a_0 : A)(a_1 : A) \to A^{\mathsf{P}}\, a_0\, a_1 \to B^{\mathsf{P}}\, (f_0\, a_0)\, (f_1\, a_1)$$

$$\mathrm{Type}^{\mathsf{P}}\, A_0\, A_1 = A_0 \to A_1 \to \mathrm{Type}$$

- Higher dimensional cubes appear when we iterate $-^{\mathsf{P}}$:

$$\frac{\dfrac{a_2 : A^{\mathsf{P}}\, a_0\, a_1}{{a_2}^{\mathsf{P}} : (A^{\mathsf{P}}\, a_0\, a_1)^{\mathsf{P}}\, a_2\, a_2}}{{a_2}^{\mathsf{P}} : A^{\mathsf{P}^{\mathsf{P}}}\, a_0\, a_0\, {a_0}^{\mathsf{P}}\, a_1\, a_1\, {a_1}^{\mathsf{P}}\, a_2\, a_2} \qquad \frac{\dfrac{(\lambda a.a^{\mathsf{P}}) : (a : A) \to A^{\mathsf{P}}\, a\, a}{(\lambda a.a^{\mathsf{P}})^{\mathsf{P}}\, a_0\, a_1\, a_2 : (A^{\mathsf{P}}\, a\, a)^{\mathsf{P}}\, {a_0}^{\mathsf{P}}\, {a_1}^{\mathsf{P}}}}{(\lambda a.a^{\mathsf{P}})^{\mathsf{P}}\, a_0\, a_1\, a_2 : A^{\mathsf{P}^{\mathsf{P}}}\, a_0\, a_1\, a_2\, a_0\, a_1\, a_2\, {a_0}^{\mathsf{P}}\, {a_1}^{\mathsf{P}}}$$

$$\begin{array}{ccc} a_1 & \xrightarrow{{a_1}^{\mathsf{P}}} & a_1 \\ {\scriptstyle a_2}\uparrow & & \uparrow{\scriptstyle a_2} \\ a_0 & \xrightarrow{{a_0}^{\mathsf{P}}} & a_0 \end{array} \qquad \begin{array}{c} a_1 \\ {\scriptstyle a_2}\uparrow \\ a_0 \end{array} \qquad \begin{array}{ccc} a_0 & \xrightarrow{a_2} & a_1 \\ {\scriptstyle {a_0}^{\mathsf{P}}}\uparrow & & \uparrow{\scriptstyle {a_1}^{\mathsf{P}}} \\ a_0 & \xrightarrow{a_2} & a_1 \end{array}$$

Dealing with higher dimensional cubes

paper	substructural interval	model	rel/span
Bernardy–Moulin (2012)	no		rel
Bernardy–Moulin (2013)	yes		rel
Bernardy–Coquand–Moulin (2015) Reboullet (2024)	yes	Reedy fibrant presheaf	rel
Nuyts–Devriese (2018)	yes	ordinary presheaf	rel
Cavallo–Harper (2021)	yes	ordinary presheaf	rel
Altenkirch–Chamoun–Kaposi–Shulman (2024)	no	ordinary presheaf	span

- Relation vs. span

  $A^{\mathsf{P}} : A \to A \to \mathsf{Type}$ vs. $\forall A : \mathsf{Type}$ together with $A \xleftarrow{0_A} \forall A \xrightarrow{1_A} A$

- In this talk we define a relation-based variant of the ACKS theory.

An excerpt of the rules

$$\frac{B : \mathsf{Type}}{\begin{array}{l}\forall B : \mathsf{Type} \\ k_B : \forall B \to B\end{array}} \qquad \frac{A : B \to \mathsf{Type} \qquad b_2 : \forall B}{\mathsf{Br}_A\, b_2 : A\,(0_B\, b_2) \to A\,(1_B\, b_2) \to \mathsf{Type}}$$

- like Bernardy–Jansson–Paterson (2010)
- like cubical type theory: "$\forall B = \mathbb{I} \to B$"

$$\frac{f : A \to B}{\mathsf{ap}\, f : \forall A \to \forall B} \qquad \frac{a : (b_x : B) \to A\, b_x}{\mathsf{apd}\, a : (b_2 : \forall B) \to \mathsf{Br}_A\, b_2\,(a\,(0_B\, b_2))\,(a\,(1_B\, b_2))}$$

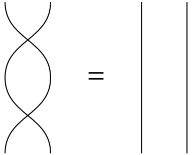
$$\frac{A : \mathsf{Type}}{\mathsf{R}_A : A \to \forall A} \qquad \frac{A : \mathsf{Type}}{\mathsf{S}_A : \forall(\forall A) \to \forall(\forall A)}$$

Some equations on k, R, S

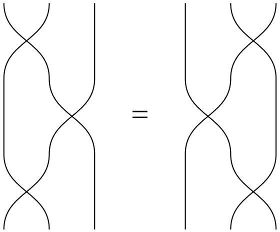
$$\overline{k_A\,(\mathsf{R}_A\,a) = a}$$

$$\overline{\mathsf{S}_A\,(\mathsf{R}_{\forall A}\,a_2) = \mathsf{ap}\,\mathsf{R}_A\,a_2}$$

$$\overline{\mathsf{S}_A\,(\mathsf{S}_A\,a_{22}) = a_{22}}$$



$$\overline{\mathsf{S}_{\forall A}\,\big(\mathsf{ap}\,\mathsf{S}_A\,(\mathsf{S}_{\forall A}\,a_{222})\big) = \mathsf{ap}\,\mathsf{S}_A\,\big(\mathsf{S}_{\forall A}\,(\mathsf{ap}\,\mathsf{S}_A\,a_{222})\big)}$$



Computing Bridge

$$\frac{}{\mathrm{Br}_{A\circ f}\, c_2 \;=\; \mathrm{Br}_A\,(\mathrm{ap}\, f\, c_2)}$$

$$\frac{}{\begin{aligned}\mathrm{Br}_{\lambda c.(a:A\,c)\to B\,(c,a)}\, c_2\, f_0\, f_1 \;\;\cong\;\; & (a_0 : A\,(0\,c_2))(a_1 : A\,(1\,c_2))(a_2 : \mathrm{Br}_A\, c_2\, a_0\, a_1) \to \\ & \mathrm{Br}_B\,(c_2, a_2)\,(f_0\, a_0)\,(f_1\, a_1)\end{aligned}}$$

$$\frac{}{\mathrm{Br}_{\lambda X.X} : \mathrm{Br}_{\lambda\_.\mathrm{Type}} * A_0\, A_1 \;\;\leftrightarrow\;\; A_0 \to A_1 \to \mathrm{Type} : \mathrm{Gel}}$$

$$\frac{}{\mathrm{gel} : R\, a_0\, a_1 \cong \mathrm{Br}_{\lambda X.X}\,(\mathrm{Gel}\, R)\, a_0\, a_1 : \mathrm{ungel}}$$

Polymorphic identity example

Assume

$$f : \big(w : \Sigma(A : \mathsf{Type}).A\big) \to \pi_1\, w,$$

$$A : \mathsf{Type},$$

$$P : A \to \mathsf{Type},$$

$$a : A,$$

$$p : P\, a,$$

using the unary version of our theory, we have

$$\mathsf{ungel}\big(\mathsf{apd}\, f\ (A, \mathsf{Gel}\, P, a, \mathsf{gel}\, p)\big) : f\ (A, a) = a.$$

when setting

$$P := \lambda x.x = a \qquad \text{and} \qquad p := \mathsf{refl},$$

Summary

- A type theory with internal parametricity:
  - relational (indexed)
  - there is no substructural interval
  - parametricity relations compute up to isomorphism, except for the universe:
    - up to weak section-retraction
- We use telescopes for $\forall$, see the abstract for details.
- Any model of our previous span-based theory is a model of the relation-based theory.
- Implementation: `github.com/mikeshulman/narya`
- Future work:
  - stricter theory
  - H.O.T.T.: adding transport