Quotient inductive-inductive types in the setoid model

Ambrus Kaposi and Zongpu Xie

Eötvös Loránd University, Budapest

TYPES 2021
online (Leiden)
14 June 2021

The setoid model

A closed type is a setoid: a set with an equivalence relation.

The setoid model justifies (Hofmann's thesis 1995)

- function extensionality
- propositional extensionality
- quotient types

It is a strict model in an intensional metatheory with SProp (Altenkirch LICS 1999).

- model construction
- setoid type theory
- Agda has SProp!

Goal of this talk: show that the setoid model justifies QIITs.

Quotient inductive-inductive types (QIITs)

Inductive types with possibly

- multiple sorts
- indexed over each other
- equality constructors

Examples:

- constructive ordinals (Kraus,Nordvall Forsberg,Chuangjie TODAY)
- Cauchy reals
- partiality monad
- syntax of type theory
- initial object for a generalised algebraic theory (Cartmell 1986) / essentially algebraic theory (Freyd 1972)

More constructive than quotients: avoid axiom of choice.

Example QIIT

$$
\begin{aligned}
\mathsf{Con} &: \mathsf{Set} \\
\mathsf{Ty} &: \mathsf{Con} \to \mathsf{Set} \\
\bullet &: \mathsf{Con} \\
- \rhd - &: (\gamma : \mathsf{Con}) \to \mathsf{Ty}\ \gamma \to \mathsf{Con} \\
\mathsf{U} &: \mathsf{Ty}\ \gamma \\
\mathsf{El} &: \mathsf{Ty}\ (\gamma \rhd \mathsf{U}) \\
\Sigma &: (a : \mathsf{Ty}\ \gamma) \to \mathsf{Ty}\ (\gamma \rhd a) \to \mathsf{Ty}\ \gamma \\
\mathsf{eq} &: \gamma \rhd \Sigma\ a\ b = \gamma \rhd a \rhd b
\end{aligned}
$$

When does a model support ...

1. ... inductive types?
   - Inductive types can be reduced to W types (in ITT)
     (Jasper Hugunin: Why not W? TYPES 2020)
2. indexed / mutual inductive types?
   - Indexed W types can be reduced to W types (in ITT)
3. IITs?
   - Finitary IITs can be reduced to indexed W types (in ETT)
     (Kaposi,Kovács,Lafont TYPES 2019)
4. QIITs?
   - QIITs can be reduced to a universal QIIT (in ETT)
     (Kaposi,Kovács LICS 2020)

Computation rules?

Conservativity of ETT over ITT+funext+UIP
(Hofmann 1995, Winterhalter,Sozeau,Tabareau 2019):

$$\frac{\Gamma \vdash_{\mathsf{ITT}} A \qquad \Gamma \vdash_{\mathsf{ETT}} t : A}{\Gamma \vdash_{\mathsf{ITT}} t' : A}$$

For any QIIT Ω, the type

"the universal QIIT exists $\Rightarrow$ Ω exists"

can be expressed in ITT.

If we prove that the setoid model supports the universal QIIT, then it supports all QIITs with propositional computation rules.

The universal QIIT

is a syntax for a small type theory:

- CwF
- U,El
- Π with domain in U
- Id with reflection
- Π with metatheoretic domain
- U is closed under Π with metatheoretic domain

4 sorts, 19 operators, 22 equations.

Example QIIT

$$
\begin{aligned}
\mathsf{Con} \;&: \mathsf{Set} \\
\mathsf{Ty} \;&: \mathsf{Con} \to \mathsf{Set} \\
\bullet \;&: \mathsf{Con} \\
- \triangleright - \;&: (\gamma : \mathsf{Con}) \to \mathsf{Ty}\ \gamma \to \mathsf{Con} \\
\mathsf{U} \;&: \mathsf{Ty}\ \gamma \\
\mathsf{El} \;&: \mathsf{Ty}\ (\gamma \triangleright \mathsf{U}) \\
\Sigma \;&: (a : \mathsf{Ty}\ \gamma) \to \mathsf{Ty}\ (\gamma \triangleright a) \to \mathsf{Ty}\ \gamma \\
\mathsf{eq} \;&: \gamma \triangleright \Sigma\ a\ b = \gamma \triangleright a \triangleright b
\end{aligned}
$$

The implementation IIT

$$
\begin{aligned}
&|\mathsf{Con}| &&: \mathsf{Set}\\
&- \sim_{\mathsf{Con}} - &&: |\mathsf{Con}| \to |\mathsf{Con}| \to \mathsf{SProp}\\
&|\mathsf{Ty}| &&: |\mathsf{Con}| \to \mathsf{Set}\\
&\sim_{\mathsf{Ty}} &&: \gamma \sim_{\mathsf{Con}} \gamma' \to |\mathsf{Ty}|\,\gamma \to |\mathsf{Ty}|\,\gamma' \to \mathsf{SProp}\\
&|\bullet| &&: |\mathsf{Con}|\\
&\sim_{\bullet} &&: |\bullet| \sim_{\mathsf{Con}} |\bullet|\\
&-|\triangleright|- &&: (\gamma : |\mathsf{Con}|) \to |\mathsf{Ty}|\ \gamma \to |\mathsf{Con}|\\
&- \sim_{\triangleright} - &&: (\bar{\gamma} : \gamma \sim_{\mathsf{Con}} \gamma') \to \sim_{\mathsf{Ty}}\ \bar{\gamma}\ \alpha\ \alpha' \to (\gamma\,|\triangleright|\,\alpha) \sim_{\mathsf{Con}} (\gamma'\,|\triangleright|\,\alpha')\\
&\dots\\
&|\mathsf{eq}| &&: \gamma\,|\triangleright|\,|\Sigma|\ a\ b \sim_{\mathsf{Con}} \gamma\,|\triangleright|\,a\,|\triangleright|\,b
\end{aligned}
$$

$\sim_{\mathsf{Con}}, \sim_{\mathsf{Ty}}$ are reflexive, symmetric and transitive

$$
\begin{aligned}
&\mathsf{coe}_{\mathsf{Ty}} &&: \gamma \sim_{\mathsf{Con}} \gamma' \to |\mathsf{Ty}|\ \gamma \to |\mathsf{Ty}|\ \gamma'\\
&\mathsf{coh}_{\mathsf{Ty}} &&: (\bar{\gamma} : \gamma \sim_{\mathsf{Con}} \gamma')(\alpha : |\mathsf{Ty}|\ \gamma) \to \sim_{\mathsf{Ty}}\ \bar{\gamma}\ \alpha\ (\mathsf{coe}_{\mathsf{Ty}}\ \bar{\gamma}\ \alpha)
\end{aligned}
$$

The QIIT in the setoid model

1. The type formation rules and constructors are defined in the empty context using the implemention IIT.
2. The recursor for a Con-Ty algebra *in the empty context* is defined by recursion-recursion on this IIT.
3. An algebra in an arbitrary context is turned into an algebra in the empty context, i.e. we turn a type $\Gamma \vdash C$ into a type $\cdot \vdash \Pi(x : \mathsf{K}\ \Gamma).C[x]$
4. Now we can use the recursor in any context. Its β rules are definitional. Its substitution laws are proven by induction-induction on the IIT.
5. Uniqueness is proved by induction-induction on the IIT.

For Con-Ty, we formalised this in Agda.
For the universal QIIT, we formalised steps 1,2,5 in Agda.

Summary

- The setoid model is poor man's cubical model.
- QIITs: free generalised algebraic theories.
- QIITs with propositional computation rules can be reduced to a universal QIIT.
- The universal QIIT can be defined in the setoid model (WIP: lifting to arbitrary contexts).
- Further work: give a direct definition of a QIIT in the setoid model for an arbitrary signature.
  This would give definitional β rules.