Induction-induction Part 3

Constructing inductive-inductive types using a domain-specific type theory[1]

Ambrus Kaposi

j.w.w. Thorsten Altenkirch, Péter Diviánszky and András Kovács

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1. Another specification of IITs
   - more syntactic than Fredrik's

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1. Another specification of IITs
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2. Constructing all IITs
   - from a universal IIT
   - conjecture:

$$\text{inductive types} \xrightarrow{\text{Jakob method}} \text{universal IIT} \xrightarrow{\text{this talk}} \text{any IIT}$$

Another specification of IITs

How to specify inductive types?

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By listing their constructors.

What does it mean to list the constructors?

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- Variables
- Empty universe U with underline for El:

$$\frac{}{\Gamma \vdash \mathsf{U}} \qquad \frac{\Gamma \vdash a : \mathsf{U}}{\Gamma \vdash \underline{a}}$$

- Restricted function space:

$$\frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma, x : \underline{a} \vdash B}{\Gamma \vdash (x : a) \Rightarrow B} \qquad \frac{\Gamma \vdash t : (x : a) \Rightarrow B \qquad \Gamma \vdash u : \underline{a}}{\Gamma \vdash t\, u : B[x \mapsto u]}$$

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Signature for natural numbers:
$\Theta := (\cdot,\ Nat : \mathsf{U},\ zero : \underline{Nat},\ suc : Nat \Rightarrow \underline{Nat})$

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$\Theta := (\cdot, \mathit{Nat} : \mathsf{U}, \mathit{zero} : \underline{\mathit{Nat}}, \mathit{suc} : \mathit{Nat} \Rightarrow \underline{\mathit{Nat}})$

Not possible: $(\cdot, T : \mathsf{U}, \mathit{evil} : (T \Rightarrow \underline{T}) \Rightarrow \underline{T})$

Standard model

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{A}} \in \mathsf{Set}} \qquad \frac{\Gamma \vdash A}{A^{\mathsf{A}} \in \Gamma^{\mathsf{A}} \to \mathsf{Set}} \qquad \frac{\Gamma \vdash t : A}{t^{\mathsf{A}} \in (\gamma \in \Gamma^{\mathsf{A}}) \to A^{\mathsf{A}}(\gamma)}$$

$$\begin{aligned}
(\Gamma, x : A)^{\mathsf{A}} &:= (\gamma \in \Gamma^{\mathsf{A}}) \times A^{\mathsf{A}}(\gamma) \\
\mathsf{U}^{\mathsf{A}}(\gamma) &:= \mathsf{Set} \\
((x : a) \Rightarrow B)^{\mathsf{A}}(\gamma) &:= (\alpha \in a^{\mathsf{A}}(\gamma)) \to B^{\mathsf{A}}(\gamma, \alpha)
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$-^{\mathsf{A}}$ on a context gives algebras for that signature.
E.g. $\Theta^{\mathsf{A}} = (N \in \mathsf{Set}) \times N \times (N \to N)$

Logical relation interpretation

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{M}} \in \Gamma^{\mathsf{A}} \to \Gamma^{\mathsf{A}} \to \mathsf{Set}} \qquad \frac{\Gamma \vdash A}{A^{\mathsf{M}} \in \Gamma^{\mathsf{M}}\, \gamma\, \gamma' \to A^{\mathsf{A}}\, \gamma \to A^{\mathsf{A}}\, \gamma' \to \mathsf{Set}}$$

$$\begin{aligned}
(\Gamma, x : A)^{\mathsf{M}}((\gamma, \alpha), (\gamma', \alpha')) &:= (\gamma_M : \Gamma^{\mathsf{M}}(\gamma, \gamma')) \times A^{\mathsf{M}}(\gamma_M, \alpha, \alpha') \\
\mathsf{U}^{\mathsf{M}}(\gamma_M, a, a') &:= a \to a' \to \mathsf{Set} \\
(\underline{a})^{\mathsf{M}}(\gamma_M, \alpha, \alpha') &:= a^{\mathsf{M}}(\gamma_M, \alpha, \alpha') \\
((x : a) \Rightarrow B)^{\mathsf{M}}(\gamma_M, f, f') &:= (\alpha_M \in a^{\mathsf{M}}(\gamma, \alpha, \alpha')) \to \\
&\qquad B^{\mathsf{M}}\Big((\gamma_M, \alpha_M), f(\alpha), f'(\alpha')\Big)
\end{aligned}$$

Tweaked logical relation interpretation

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\end{aligned}$$

$-^{\mathsf{M}}$ on a context gives homomorphisms of algebras. E.g.

$$\begin{aligned}
\Theta^{\mathsf{M}}((N,z,s),(N',z',s')) = (N_M : N \to N') \times (N_M(z) = z')\times \\
((\alpha \in N) \to N_M(s(\alpha)) = s'(N_M(\alpha)))
\end{aligned}$$

Constructing inductive-inductive types

Constructing natural numbers

We use the domain-specific type theory:

$$\mathbb{N} := \{t \mid \Theta \vdash t : \underline{Nat}\}$$

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Constructors:

$$\mathsf{zero} := \mathit{zero} \in \mathbb{N}$$

$$\mathsf{suc}(n \in \mathbb{N}) := \mathit{suc}\ n \in \mathbb{N}$$

Recursor for $\mathbb{N}$

We need:

$$\mathsf{rec}_{\mathbb{N}} : \mathbb{N} \to (x : \Theta^{\mathsf{A}}) \to \mathsf{proj}_1(x)$$

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The standard interpretation of $t \in \mathbb{N}$ i.e. $\Theta \vdash t : \underline{Nat}$:

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$$\mathsf{rec}_{\mathbb{N}}(t) := t^{\mathsf{A}}$$

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Fix a Θ.

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Initial algebra for Θ is $\Theta^{\mathsf{C}}(\mathrm{id}_\Theta) \in \Theta^{\mathsf{A}}$.

Model for the recursor

Fix a Θ and a $\theta \in \Theta^{\mathsf{A}}$.

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{R}} \in (\Theta \vdash \nu : \Gamma) \to \Gamma^{\mathsf{M}}(\Gamma^{\mathsf{C}}(\nu), \nu^{\mathsf{A}}(\theta))}$$

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Fix a Θ and a $\theta \in \Theta^{\mathsf{A}}$.

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Recursor is given by $\Theta^{\mathsf{R}}(\mathsf{id}_\Theta) \in \Theta^{\mathsf{M}}(\text{initial algebra for } \Theta, \theta)$.

Summary

Domain-specific type theory for signatures.

We do universal algebra by defining models of this type theory.

Standard model:	algebras
Tweaked logical relations:	algebra homomorphisms
Model where U is terms:	initial algebra
Model where U is the standard interpretation:	recursor
Logical predicates:	families
Tweaked dependent logical relations:	sections
Model where U is logical predicate translation:	eliminator

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It seems that this extends to quotient inductive-inductive types.
Challenge: higher inductive-inductive types?

CwF of CwFs

standard model	algebras	contexts
tweaked logical relations	homomorphisms	substitutions
logical predicates	families	types
tweaked dependent logical relations	sections	terms
reflexivity in the reflexive graph model	identity morphism	identity substitution
transitivity in the setoid model	morphisms composition	substitutions composition
groupoid law in the groupoid model	associativity	substitution law

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Nice internal language for universal algebra.

Constructing inductive types à la Awodey-Frey-Speight

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Solution: let's build this in the definition!

$$\mathbb{N} := (n \in (x : \Theta^{\mathsf{A}}) \to \mathsf{proj}_1(x)) \times (\forall x, x', f. f(n(x)) = n(x'))$$