Normalisation by Evaluation for Dependent Types

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Introduction

- Goal:
  - Prove normalisation for a type theory with dependent types
  - Using the metalanguage of type theory itself
- Structure of the talk:
  - Representing type theory in type theory
  - Specifying normalisation
  - NBE for simple types
  - NBE for dependent types

Representing type theory in type theory

Simple type theory in idealised Agda

```
data Ty  : Set where
  ι      : Ty
  _⇒_    : Ty → Ty → Ty
data Con : Set where
  •      : Con
  _,_    : Con → Ty → Con
data Var : Con → Ty → Set where
  zero   : Var (Γ , A) A
  suc    : Var Γ A → Var (Γ , B) A
data Tm  : Con → Ty → Set where
  var    : Var Γ A → Tm Γ A
  lam    : Tm (Γ , A) B → Tm Γ (A ⇒ B)
  app    : Tm Γ (A ⇒ B) → Tm Γ A → Tm Γ B
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```

No preterms!

A typed syntax of dependent types (i)

- Types depend on contexts.
  $\Rightarrow$ We need induction induction.

$$\begin{array}{lll} \textbf{data}\ \mathsf{Con} & : & \mathsf{Set} \\ \textbf{data}\ \mathsf{Ty} & : & \mathsf{Con} \to \mathsf{Set} \end{array}$$

A typed syntax of dependent types (ii)

- Types depend on contexts.
  $\Rightarrow$ We need induction induction.
- Substitutions are mentioned in the application rule:

$$\mathsf{app} : \mathsf{Tm}\,\Gamma\,(\Pi\,A\,B) \to (a : \mathsf{Tm}\,\Gamma\,A) \to \mathsf{Tm}\,\Gamma\,(B[a])$$

  $\Rightarrow$ We define an explicit substitution calculus.

```
data Con  : Set
data Ty   : Con → Set
data Tms  : Con → Con → Set
data Tm   : (Γ : Con) → Ty Γ → Set
  _[_] : Ty Γ → Tms Δ Γ → Ty Δ
  ...
```

A typed syntax of dependent types (iii)

- Types depend on contexts.
  $\Rightarrow$ We need induction induction.
- Substitutions are mentioned in the application rule:
  $\Rightarrow$ We define an explicit substitution calculus.
- The following conversion rule for terms:

$$\frac{\Gamma \vdash A \sim B \qquad \Gamma \vdash t : A}{\Gamma \vdash t : B}$$

  $\Rightarrow$ Conversion (the relation including β, η) needs to be defined mutually with the syntax.
  - We need to add 4 new members to the inductive inductive definition: $\sim$ for contexts, types, substitutions and terms.

Representing conversion

- Lots of boilerplate:
  - The $\sim$ relations are equivalence relations
  - Coercion rules
  - Congruence rules
  - We need to work with setoids
- The identity type $\_\equiv\_$ is an equivalence relation with coercion and congruence laws.
- Higher inductive types are an idea from homotopy type theory: constructors for equalities.
- We add the conversion rules as constructors: e.g. $\beta : \mathsf{app}\,(\mathsf{lam}\,t)\,u \equiv t[u]$.

QIITs

We formalise the syntax of type theory as a quotient inductive inductive type (QIIT).

- A QIT is a HIT which is a set
- QITs are not the same as quotient types

Using the syntax

- One defines functions from a QIIT using its eliminator.
- The arguments of the non-dependent eliminator form a model of type theory, equivalent to Categories with Families.

$$
\begin{array}{lll}
\textbf{record}\ \mathsf{Model} & : & \mathsf{Set}\ \textbf{where} \\
\quad \textbf{field}\ \mathsf{Con}^{\mathsf{M}} & : & \mathsf{Set} \\
\quad\quad \mathsf{Ty}^{\mathsf{M}} & : & \mathsf{Con}^{\mathsf{M}} \to \mathsf{Set} \\
\quad\quad \mathsf{Tm}^{\mathsf{M}} & : & (\Gamma : \mathsf{Con}^{\mathsf{M}}) \to \mathsf{Ty}^{\mathsf{M}}\ \Gamma \to \mathsf{Set} \\
\quad\quad \mathsf{lam}^{\mathsf{M}} & : & \mathsf{Tm}^{\mathsf{M}}\ (\Gamma\ ,^{\mathsf{M}}\ \mathsf{A})\ \mathsf{B}^{\mathsf{M}} \to \mathsf{Tm}^{\mathsf{M}}\ \Gamma\ (\Pi^{\mathsf{M}}\ \mathsf{A}\ \mathsf{B}) \\
\quad\quad \beta^{\mathsf{M}} & : & \mathsf{app}^{\mathsf{M}}\ (\mathsf{lam}^{\mathsf{M}}\ \mathsf{t})\ \mathsf{a} \equiv \mathsf{t}\ [\ \mathsf{a}\ ]^{\mathsf{M}} \\
\quad\quad \ldots &&
\end{array}
$$

- The eliminator says that the syntax is the initial model.

Specifying normalisation

Specifying normalisation

Neutral terms and normal forms (typed!):

$$\begin{array}{lll} \mathsf{n} ::= \mathsf{x} \mid \mathsf{n}\ \mathsf{v} & & \mathsf{Ne}\ \Gamma\ A \\ \mathsf{v} ::= \mathsf{n} \mid \lambda\, \mathsf{x}\, .\mathsf{v} & & \mathsf{Nf}\ \Gamma\ A \end{array}$$

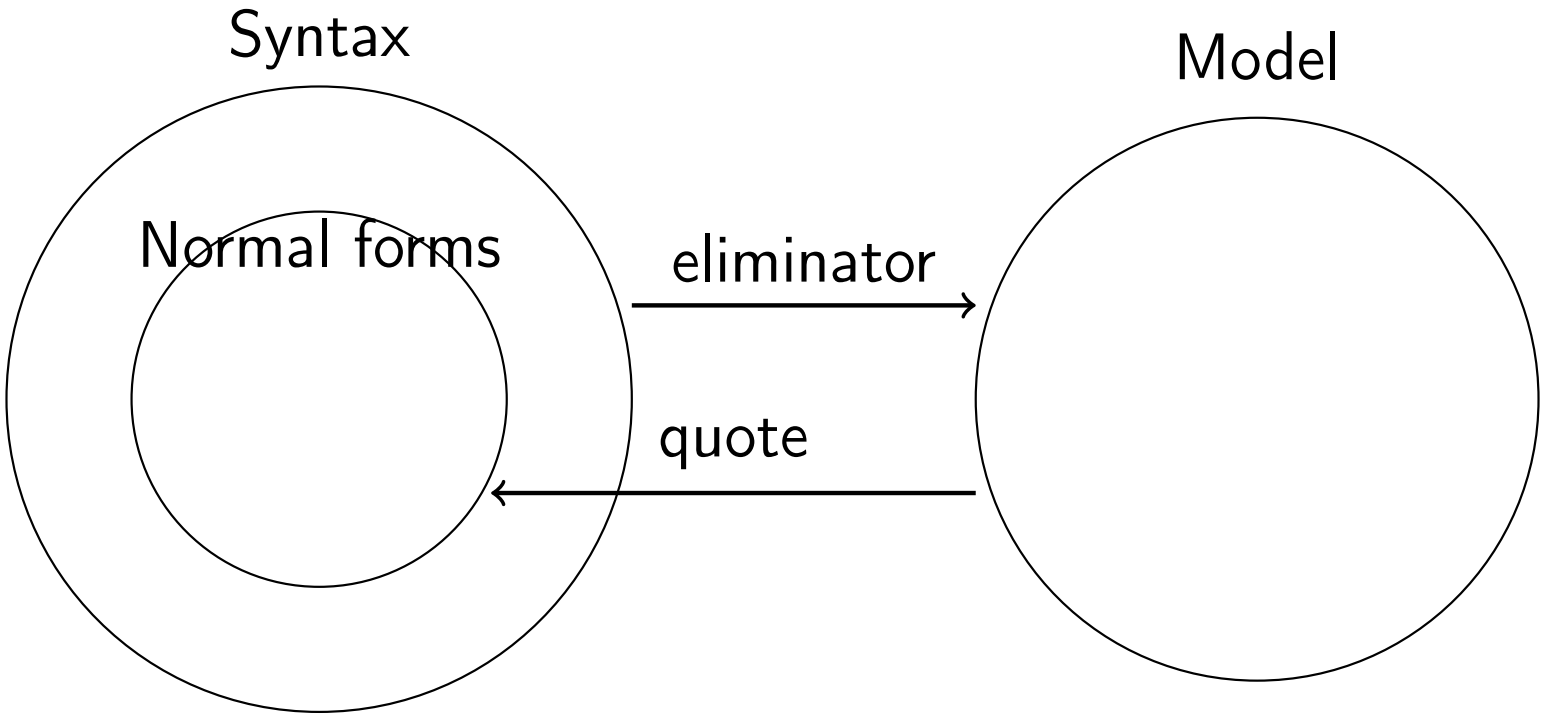
Normalisation is an isomorphism:

$$\text{completeness} \circlearrowleft \quad \text{norm} \downarrow \quad \frac{\mathsf{Tm}\ \Gamma\ A}{\mathsf{Nf}\ \Gamma\ A} \quad \uparrow^{\ulcorner - \urcorner} \quad \curvearrowright \text{stability}$$

Soundness is given by congruence of equality:

$$t \equiv t' \to \mathsf{norm}\ t \equiv \mathsf{norm}\ t'$$

Normalisation by Evaluation (NBE)



- First formulation (Berger and Schwichtenberg, 1991)
- Simply typed case (Altenkirch, Hofmann, Streicher 1995)
- Dependent types using untyped realizers (Abel, Coquand, Dybjer, 2007)

NBE for simple types

The presheaf model

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  - Given another context Δ we have $[\![\Gamma]\!]_\Delta : \mathrm{Set}$.
  - Given a renaming $\Delta \xrightarrow{\beta} \Theta$, there is a $[\![\Gamma]\!]_\Theta \xrightarrow{[\![\Gamma]\!]\,\beta} [\![\Gamma]\!]_\Delta$.

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Quotation

The quote function is a natural transformation.

$$\mathsf{quote}_A : [\![A]\!] \dot{\to} \mathsf{Nf} - A$$

At a given context we have:

$$\mathsf{quote}_{A\,\Gamma} : [\![A]\!]_\Gamma \;\to\; \mathsf{Nf}\,\Gamma\,A$$

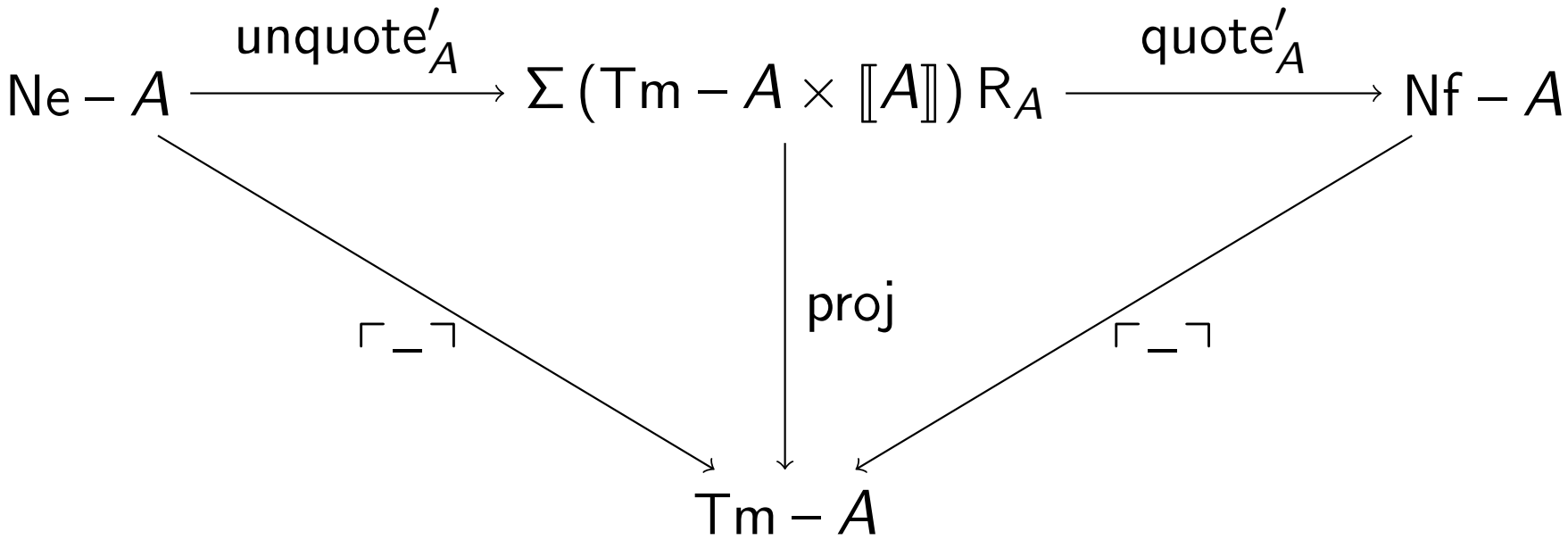
It is defined mutually with unquote:

$$\mathsf{unquote}_A : \mathsf{Ne} - A \dot{\to} [\![A]\!]$$

Quote and unquote

$$\mathsf{Ne} - A \xrightarrow{\text{unquote}\ A} [\![A]\!] \xrightarrow{\text{quote}\ A} \mathsf{Nf} - A$$

With completeness



R_A is a presheaf logical relation between the syntax and the presheaf model. It is equality at the base type.

NBE for dependent types

Defining quote, first try

$$\mathsf{Nes} - \Gamma \xrightarrow{\mathsf{unquote}_\Gamma} [\![\Gamma]\!] \xrightarrow{\mathsf{quote}_\Gamma} \mathsf{Nfs} - \Gamma$$

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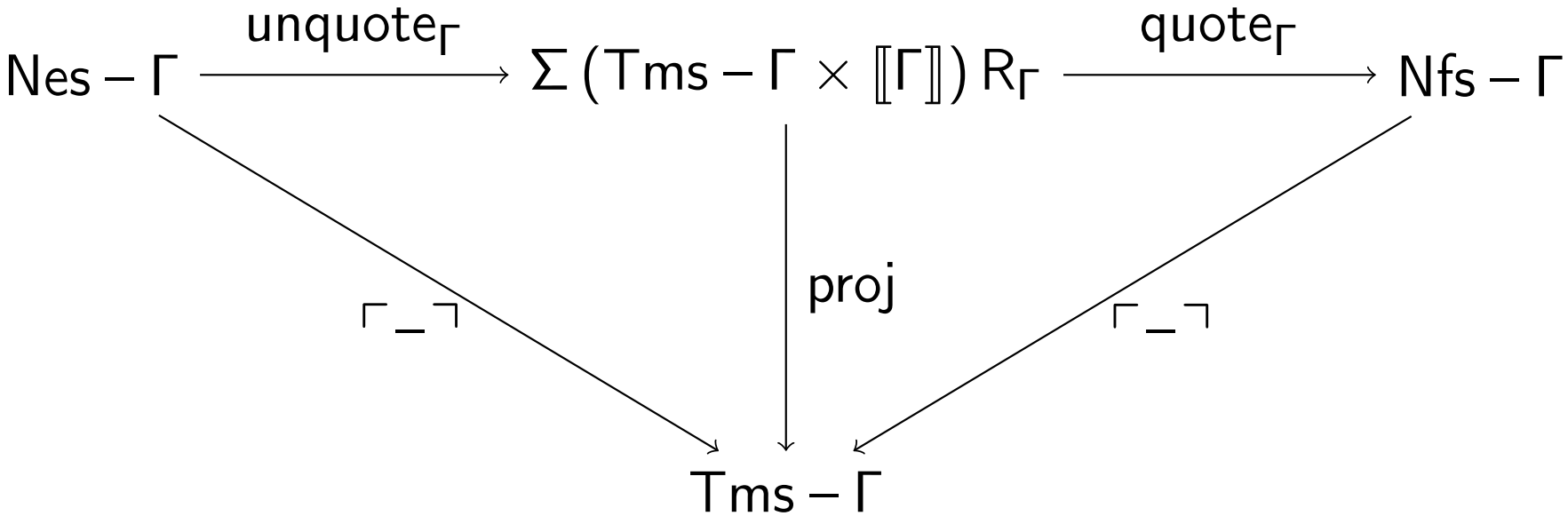
Let's define quote and its completeness mutually!

Defining quote, second try



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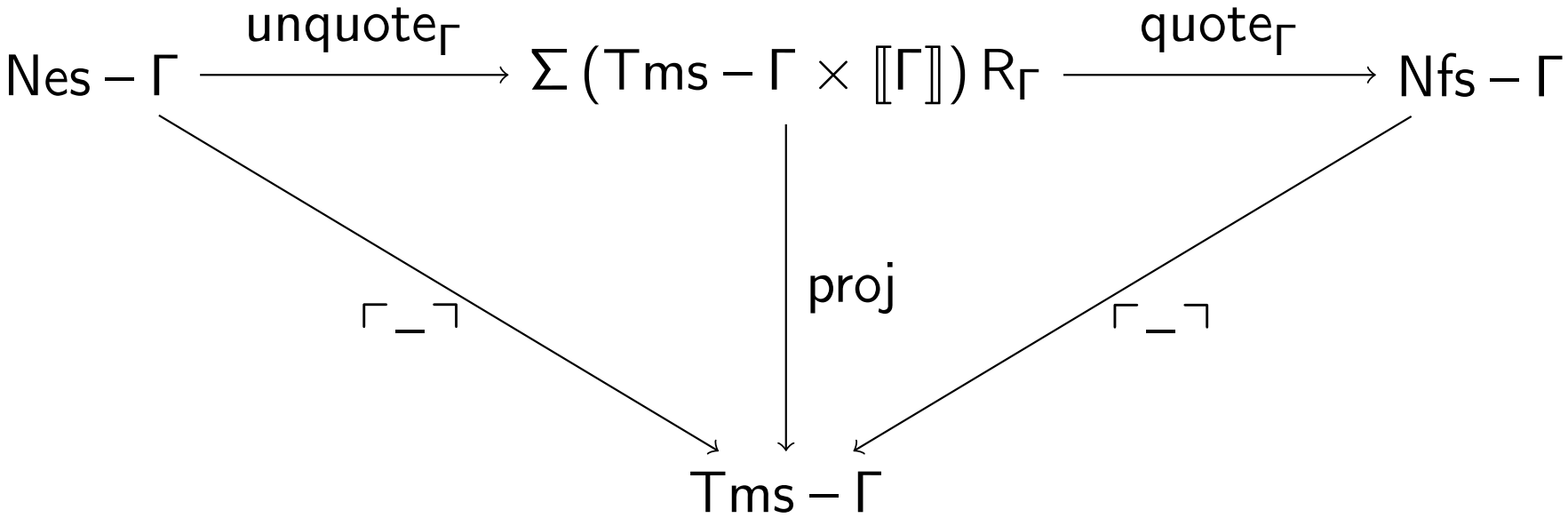
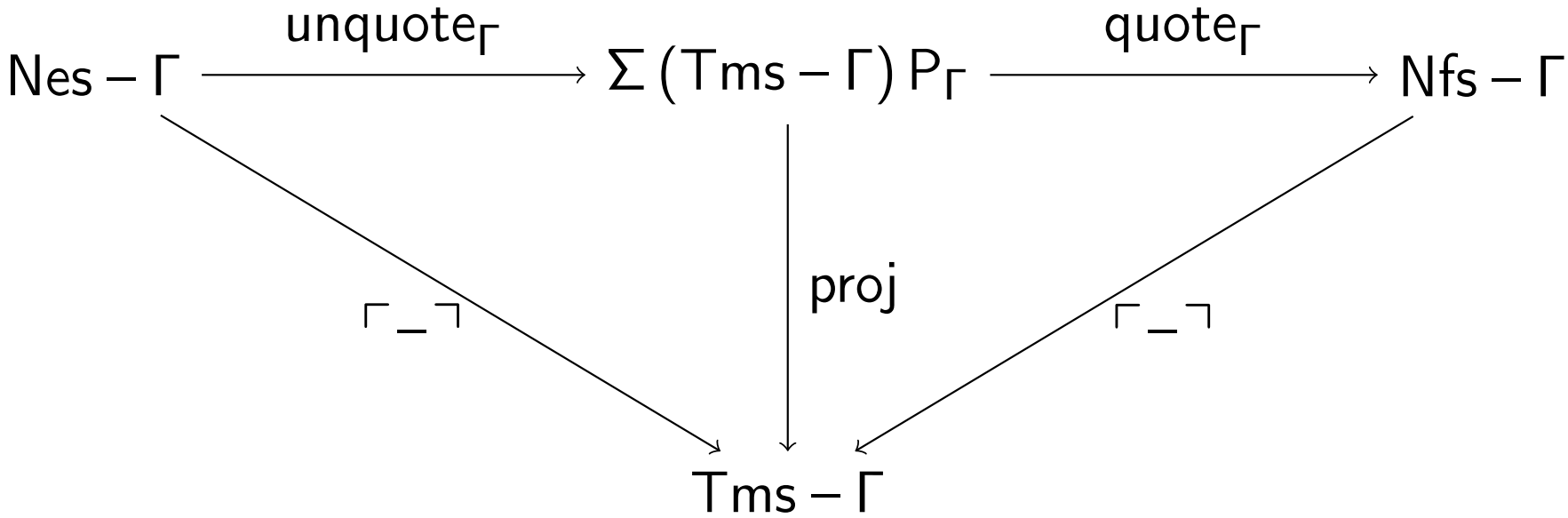


For unquote at the function space we need to define a semantic function which works for every input, not necessarily related by the relation. But quote needs ones which are related!

Defining quote, third try



Use a proof-relevant logical predicate. At the base type it says that there exists a normal form which is equal to the term.

Instance of categorical glueing.

Extra slides

The presheaf model and quote

For dependent types, types are interpreted as families of presheaves.

$$\begin{aligned} [\![\Gamma]\!] &: \mathrm{REN}^{\mathrm{op}} \to \mathrm{Set} \\ [\![\Gamma \vdash A]\!] &: (\Delta : \mathrm{REN}) \to [\![\Gamma]\!]_\Delta \to \mathrm{Set} \end{aligned}$$

The presheaf model and quote

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Quote for contexts is the same, but for types it is more subtle:

$$\begin{aligned}\mathsf{quote}_\Gamma &: \llbracket \Gamma \rrbracket \dot{\to} \mathsf{Tms} - \Gamma \\ \mathsf{quote}_{\Gamma \vdash A} &: (\alpha : \llbracket \Gamma \rrbracket_\Delta) \to \llbracket A \rrbracket_\Delta\, \alpha \to \mathsf{Nf}\, \Delta \left(A[\mathsf{quote}_{\Gamma,\Delta}\, \alpha]\right)\end{aligned}$$

Quotation

The quote function is a natural transformation.

$$\mathsf{quote}_A : [\![A]\!] \dot{\to} \mathsf{Nf} - A$$

For the base type it is the identity.

$$\mathsf{quote}_\iota\ v := v$$

For function types:

$$\mathsf{quote}_{A\to B\,\Delta}\left(f : \forall\Theta.(\beta : \Theta \to \Delta) \to [\![A]\!]_\Theta \to [\![B]\!]_\Theta\right) : \mathsf{Nf}\,\Delta\,(A \to B)$$

$$:= \mathsf{lam}\left(\right.$$

$$\uparrow \mathsf{Nf}\,(\Delta, A)\,B$$

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$$\uparrow [\![B]\!]_{\Delta,A}$$

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$$\uparrow \Delta, A \to \Delta$$

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We need to unquote neutral terms: $\mathsf{unquote}_A : \mathsf{Ne} - A \dot{\to} [\![A]\!]$.

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$$:= \mathsf{lam}\left(\mathsf{quote}_{B,(\Delta,A)}\left(f_{\Delta,A}\quad \mathsf{wk}\quad (\mathsf{unquote}_{A\,(\Delta,A)}\,\mathsf{zero})\right)\right)$$

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