Type Theory in Type Theory using a Strictified Syntax

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`"(λx.x+x) 3"`

`"(λx.x+x) 3a"`

`"((λx.x+x) 3)"`

`"(λx.x + x) 3"`

`"(λx.x+x 3"`

`"6"`

`"(λy.y+y) 3"`

`"(λx.x+y) 3"`

`"(λx.x+x) true"`

string

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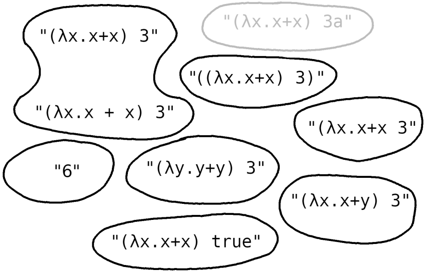

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removal of spaces lexical analysis add spaces

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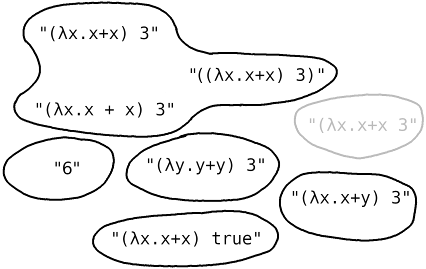

abstract syntax tree

removal of extra brackets — parsing — add brackets

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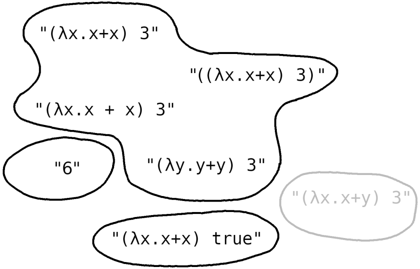

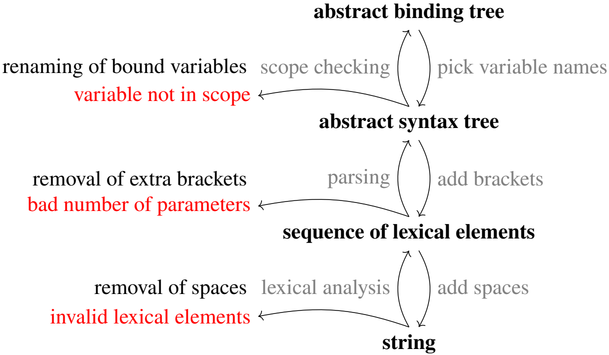

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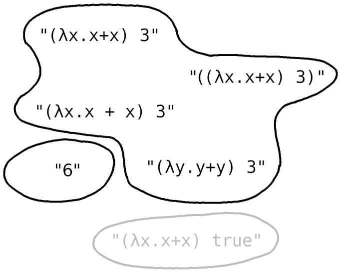

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type checking

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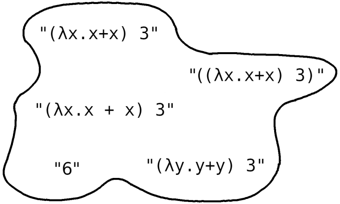

well-typed syntax quotiented by conversion

operational equivalence — normalise

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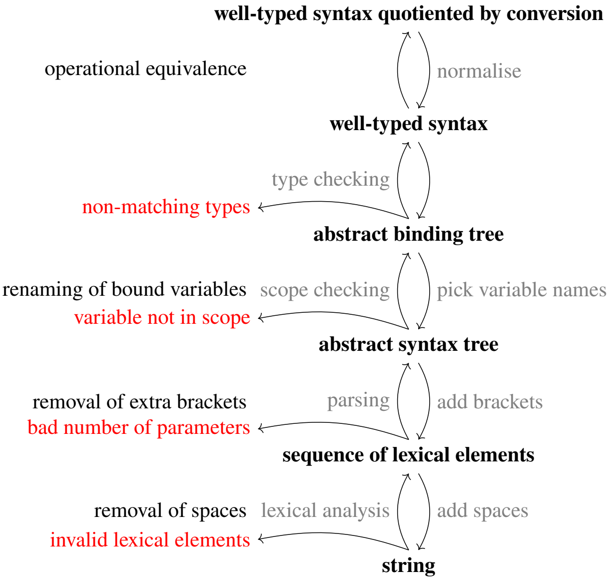

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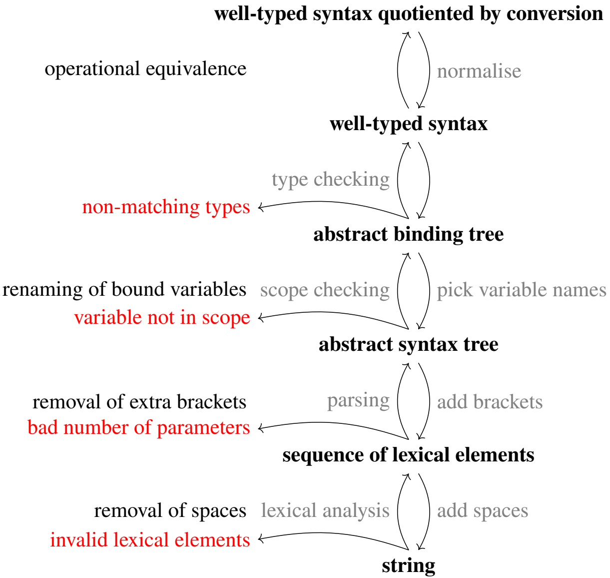

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    (gluing a.k.a. proof relevant
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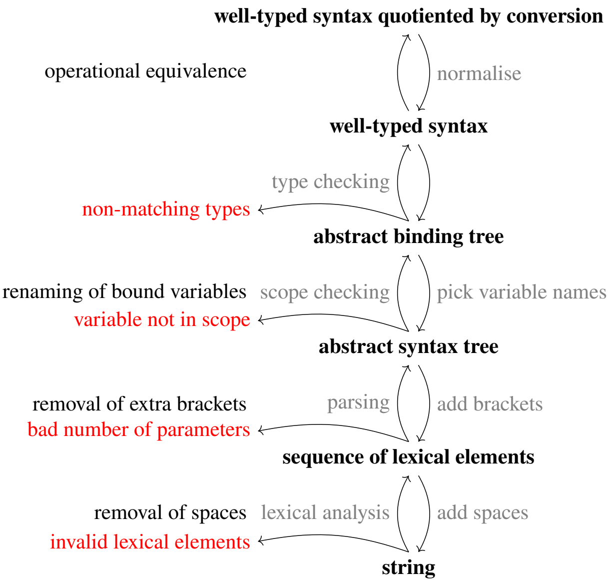

Computer formalisation

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- We formalised gluing style canoncity for a small type theory, the Agda proof is as beautiful as the paper one.

Analogs of our strictification technique

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$$(\alpha : \forall K. Hom(K, J) \to Hom(K, I)) \qquad \text{such that} \qquad (\alpha\, K\, f) \circ g = \alpha\, L\, (f \circ g).$$

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$$\mathsf{y}\,J \xrightarrow{\bullet} \mathsf{y}\,I.$$

  The Yoneda lemma implies $Hom(J, I) \;\cong\; \mathsf{y}\,J \xrightarrow{\bullet} \mathsf{y}\,I.$

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- Also called strictification ($\cong\ \to\ =$, while our method is $=\ \to\ \equiv$):
  - right adjoint splitting (Hofmann 1994)
  - left adjoint splitting, local universes (Lumsdaine–Warren 2015)

The strictification technique

- A universe closed under Σ:

$$
\begin{aligned}
\mathsf{U} &: \mathsf{Set} & \Sigma \quad &: (A : \mathsf{U}) \to (\mathsf{El}\, A \to \mathsf{U}) \to \mathsf{U} \\
\mathsf{El} &: \mathsf{U} \to \mathsf{Set} & -,- &: (a : \mathsf{El}\, A) \times \mathsf{El}\, B \cong \mathsf{El}\, (\Sigma\, A\, B) : \mathsf{fst}, \mathsf{snd}
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- Given a universe closed under certain type formers, we build a CwF closed under the same type formers inheriting the substitution calculus from the metatheory:

$$\begin{aligned} &\mathsf{Con} := \mathsf{Set} && \mathsf{Tm}\,\Gamma\,A := (\gamma : \Gamma) \to \mathsf{El}\,(A\,\gamma) \\ &\mathsf{Sub}\,\Delta\,\Gamma := \Delta \to \Gamma && A[\sigma] := A \circ \sigma \\ &\mathsf{Ty}\,\Gamma := \Gamma \to \mathsf{U} && \Sigma\,A\,B := \lambda\gamma.\,\Sigma\,(A\,\gamma)\,(\lambda a.\,B\,(\gamma, a)) \end{aligned}$$

$$(\Sigma\,A\,B)[\tau] = \lambda\delta.\,(A\,(\tau\,\delta))\,(\lambda a.\,B(\tau\,\delta, a)) = \Sigma\,(A[\tau])\,(B[\tau^{\uparrow}])$$

The strictification technique

- A universe closed under Σ:

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- This is a substitution-strict model (called *contextualisation*, standard/set/type-model).
- Internally to presheaves over a model supporting some type formers, we have a universe closed under the same type formers.
- We take its contextualisation, then externalise (we need a strict CwF_{Π} of presheaves).

Other ways to strictify

- Just work internally to an LF (Sterling PhD 2022, Bocquet–K.–Sattler FSCD 2023).

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- Local universe $\cong \longrightarrow =$ strictification also provides some $= \longrightarrow \equiv$ (Lumsdaine–Warren 2015).
- Redefining substitution recursively (K. TYPES 2023).

Summary

- A technique for strictifying ($= \longrightarrow \equiv$) the substitution calculus of a model of TT.
  - Intrinsic quotiented syntax is now even nicer than extrinsic syntax.
  - The first computer formalisation of gluing-style canonicity for type theory.

Summary

- A technique for strictifying ($= \to \equiv$) the substitution calculus of a model of TT.
  - Intrinsic quotiented syntax is now even nicer than extrinsic syntax.
  - The first computer formalisation of gluing-style canonicity for type theory.
- Problems:
  - Agda and Coq hang when trying to compute with the strictified syntax.
  - We lose some definitional computation rules for the eliminator of the syntax.

Summary

- A technique for strictifying ($= \to \equiv$) the substitution calculus of a model of TT.
  - Intrinsic quotiented syntax is now even nicer than extrinsic syntax.
  - The first computer formalisation of gluing-style canonicity for type theory.
- Problems:
  - Agda and Coq hang when trying to compute with the strictified syntax.
  - We lose some definitional computation rules for the eliminator of the syntax.
- Future work:
  - Strictify the substitution calculus for a model of any SOGAT.
  - Reusable library.
  - Direct efficient proof assistant support.

Weak CwF

$$
\begin{aligned}
\mathsf{Con} &: \mathsf{Set} \\
\mathsf{Sub} &: \mathsf{Con} \to \mathsf{Con} \to \mathsf{Set} \\
\mathsf{Ty} &: \mathsf{Con} \to \mathsf{Set} \\
\mathsf{Tm} &: (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Set} \\
- \circ - &: \mathsf{Sub}\,\Delta\,\Gamma \to \mathsf{Sub}\,\Theta\,\Delta \to \mathsf{Sub}\,\Theta\,\Gamma \\
\mathsf{ass} &: (\gamma \circ \delta) \circ \theta = \gamma \circ (\delta \circ \theta) \\
\mathsf{id} &: \mathsf{Sub}\,\Gamma\,\Gamma \\
\mathsf{idl} &: \mathsf{id} \circ \gamma = \gamma \\
\mathsf{idr} &: \gamma \circ \mathsf{id} = \gamma \\
\diamond &: \mathsf{Con} \\
\epsilon &: \mathsf{Sub}\,\Gamma\,\diamond \\
\diamond\eta &: (\sigma : \mathsf{Sub}\,\Gamma\,\diamond) \to \sigma = \epsilon \\
-[-] &: \mathsf{Ty}\,\Gamma \to \mathsf{Sub}\,\Delta\,\Gamma \to \mathsf{Ty}\,\Delta \\
[\circ] &: A[\gamma \circ \delta] = A[\gamma][\delta] \\
[\mathsf{id}] &: A[\mathsf{id}] = A
\end{aligned}
$$

$$
\begin{aligned}
-[-] &: \mathsf{Tm}\,\Gamma\,A \to (\gamma : \mathsf{Sub}\,\Delta\,\Gamma) \to \\
&\quad\ \mathsf{Tm}\,\Delta\,(A[\gamma]) \\
[\circ] &: [\circ]_*\,(a[\gamma \circ \delta]) = a[\gamma][\delta] \\
[\mathsf{id}] &: [\mathsf{id}]_*\,(a[\mathsf{id}]) = a \\
- \triangleright - &: (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Con} \\
-,- &: (\gamma : \mathsf{Sub}\,\Delta\,\Gamma) \to \mathsf{Tm}\,\Delta\,(A[\gamma]) \to \\
&\quad\ \mathsf{Sub}\,\Delta\,(\Gamma \triangleright A) \\
,\circ &: (\gamma, a) \circ \delta = (\gamma \circ \delta, [\circ]_*\,(a[\delta])) \\
\mathsf{p} &: \mathsf{Sub}\,(\Gamma \triangleright A)\,\Gamma \\
\mathsf{q} &: \mathsf{Tm}\,(\Gamma \triangleright A)\,(A[\mathsf{p}]) \\
\triangleright\beta_1 &: \mathsf{p} \circ (\gamma, a) = \gamma \\
\triangleright\beta_2 &: ([\circ] \cdot \triangleright\beta_1)_*\,(\mathsf{q}[\gamma, a]) = a \\
\triangleright\eta &: \mathsf{id} = (\mathsf{p}, \mathsf{q})
\end{aligned}
$$

Strict CwF

$$
\begin{aligned}
\mathsf{Con} &: \mathsf{Set} \\
\mathsf{Sub} &: \mathsf{Con} \to \mathsf{Con} \to \mathsf{Set} \\
\mathsf{Ty} &: \mathsf{Con} \to \mathsf{Set} \\
\mathsf{Tm} &: (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Set} \\
- \circ - &: \mathsf{Sub}\,\Delta\,\Gamma \to \mathsf{Sub}\,\Theta\,\Delta \to \mathsf{Sub}\,\Theta\,\Gamma \\
\mathsf{ass} &: (\gamma \circ \delta) \circ \theta \equiv \gamma \circ (\delta \circ \theta) \\
\mathsf{id} &: \mathsf{Sub}\,\Gamma\,\Gamma \\
\mathsf{idl} &: \mathsf{id} \circ \gamma \equiv \gamma \\
\mathsf{idr} &: \gamma \circ \mathsf{id} \equiv \gamma \\
\diamond &: \mathsf{Con} \\
\epsilon &: \mathsf{Sub}\,\Gamma\,\diamond \\
\diamond\eta &: (\sigma : \mathsf{Sub}\,\Gamma\,\diamond) \to \sigma \equiv \epsilon \\
-[-] &: \mathsf{Ty}\,\Gamma \to \mathsf{Sub}\,\Delta\,\Gamma \to \mathsf{Ty}\,\Delta \\
[\circ] &: A[\gamma \circ \delta] \equiv A[\gamma][\delta] \\
[\mathsf{id}] &: A[\mathsf{id}] \equiv A
\end{aligned}
$$

$$
\begin{aligned}
-[-] &: \mathsf{Tm}\,\Gamma\,A \to (\gamma : \mathsf{Sub}\,\Delta\,\Gamma) \to \\
&\quad\ \mathsf{Tm}\,\Delta\,(A[\gamma]) \\
[\circ] &: a[\gamma \circ \delta] \equiv a[\gamma][\delta] \\
[\mathsf{id}] &: a[\mathsf{id}] \equiv a \\
- \triangleright - &: (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Con} \\
-,- &: (\gamma : \mathsf{Sub}\,\Delta\,\Gamma) \to \mathsf{Tm}\,\Delta\,(A[\gamma]) \to \\
&\quad\ \mathsf{Sub}\,\Delta\,(\Gamma \triangleright A) \\
,\circ &: (\gamma, a) \circ \delta \equiv (\gamma \circ \delta, a[\delta]) \\
\mathsf{p} &: \mathsf{Sub}\,(\Gamma \triangleright A)\,\Gamma \\
\mathsf{q} &: \mathsf{Tm}\,(\Gamma \triangleright A)\,(A[\mathsf{p}]) \\
\triangleright\beta_1 &: \mathsf{p} \circ (\gamma, a) \equiv \gamma \\
\triangleright\beta_2 &: \mathsf{q}[\gamma, a] \equiv a \\
\triangleright\eta &: \mathsf{id} = (\mathsf{p}, \mathsf{q})
\end{aligned}
$$

Booleans in a weak CwF (i)

$$
\begin{aligned}
&\mathsf{Bool} &&: \mathsf{Ty}\,\Gamma\\
&\mathsf{Bool}[] &&: \mathsf{Bool}[\gamma] = \mathsf{Bool}\\
&\mathsf{true} &&: \mathsf{Tm}\,\Gamma\,\mathsf{Bool}\\
&\mathsf{true}[] &&: \mathsf{Bool}[]_*\,(\mathsf{true}[\gamma]) = \mathsf{true}\\
&\mathsf{false} &&: \mathsf{Tm}\,\Gamma\,\mathsf{Bool}\\
&\mathsf{false}[] &&: \mathsf{Bool}[]_*\,(\mathsf{false}[\gamma]) = \mathsf{false}\\
&\mathsf{ind} &&: (P : \mathsf{Ty}\,(\Gamma \triangleright \mathsf{Bool})) \to \mathsf{Tm}\,\Gamma\,(P[\langle \mathsf{true}\rangle]) \to \mathsf{Tm}\,\Gamma\,(P[\langle \mathsf{false}\rangle]) \to\\
& &&\quad (b : \mathsf{Tm}\,\Gamma\,\mathsf{Bool}) \to \mathsf{Tm}\,\Gamma\,(P[\langle b\rangle])\\
&\mathsf{ind}[] &&: (\alpha\,b)_*\,((\mathsf{ind}\,P\,p\,p'\,b)[\gamma]) =\\
& &&\quad \mathsf{ind}\,(\mathsf{Bool}[]_*\,(P[\gamma^{\uparrow}]))\,(\mathsf{true}[]_*\,((\alpha\,\mathsf{true})_*\,(p[\gamma])))\,(\mathsf{false}[]_*\,((\alpha\,\mathsf{false})_*\,(p'[\gamma])))\\
& &&\qquad (\mathsf{Bool}[]_*\,(b[\gamma]))\\
&\mathsf{Bool}\beta_1 &&: \mathsf{ind}\,P\,p\,p'\,\mathsf{true} = p\\
&\mathsf{Bool}\beta_2 &&: \mathsf{ind}\,P\,p\,p'\,\mathsf{false} = p'
\end{aligned}
$$

where

$$
\alpha : (u : \mathsf{Tm}\,\Gamma\,\mathsf{Bool}) \to P[\langle u\rangle][\gamma] = P[\mathsf{Bool}[]_*\,(\gamma^{\uparrow})][\langle \mathsf{Bool}[]_*\,(u[\gamma])\rangle]
$$

Booleans in a weak CwF (ii)

$$
\begin{aligned}
\alpha\, u : & P[\langle u\rangle][\gamma] && =([\circ]) \\
& P[\langle u\rangle \circ \gamma] && \equiv \\
& P[(\mathsf{id}, [\mathsf{id}]_* \, u) \circ \gamma] && =(,\circ) \\
& P[\mathsf{id} \circ \gamma, [\circ]_* \, (([\mathsf{id}]_* \, u)[\gamma])] && =(-[-] \text{ and transport}) \\
& P[\mathsf{id} \circ \gamma, [\circ]_* \, ([\mathsf{id}]_* \, (u[\gamma]))] && =(\mathsf{idl}) \\
& P[\gamma, \mathsf{idl}_* \, ([\circ]_* \, ([\mathsf{id}]_* \, (u[\gamma])))] && =(\cdot_*) \\
& P[\gamma, ([\mathsf{id}] \cdot [\circ] \cdot \mathsf{idl})_* \, (u[\gamma])] && \equiv \\
& P[\gamma, u[\gamma]] && \equiv \\
& P[\gamma, ([\mathsf{id}] \cdot [\mathsf{id}])_* \, (u[\gamma])] && =(\cdot_*) \\
& P[\gamma, [\mathsf{id}]_* \, ([\mathsf{id}]_* \, (u[\gamma]))] && =(\triangleright\beta_2) \\
& P[\gamma, [\mathsf{id}]_* \, (([\circ] \cdot \triangleright\beta_1)_* \, (\mathsf{q}[\langle u[\gamma]\rangle]))] && \equiv \\
& P[\gamma, [\mathsf{id}]_* \, (([\circ] \cdot [\circ] \cdot \mathsf{ass} \cdot \triangleright\beta_1 \cdot \mathsf{idr} \cdot [\mathsf{id}])_* \, (\mathsf{q}[\langle u[\gamma]\rangle]))] && =(\cdot_*) \\
& P[\gamma, ([\circ] \cdot [\circ] \cdot \mathsf{ass} \cdot \triangleright\beta_1 \cdot \mathsf{idr} \cdot [\mathsf{id}] \cdot [\mathsf{id}])_* \, (\mathsf{q}[\langle u[\gamma]\rangle])] && \equiv \\
& P[\gamma, ([\circ] \cdot [\circ] \cdot \mathsf{ass} \cdot \triangleright\beta_1 \cdot \mathsf{idr})_* \, (\mathsf{q}[\langle u[\gamma]\rangle])] && =(\cdot_*) \\
& P[\gamma, \mathsf{idr}_* \, (\triangleright\beta_{1*} \, (\mathsf{ass}_* \, ([\circ]_* \, ([\circ]_* \, (\mathsf{q}[\langle u[\gamma]\rangle])))))] && =(\mathsf{idr}) \\
& P[\gamma \circ \mathsf{id}, \triangleright\beta_{1*} \, (\mathsf{ass}_* \, ([\circ]_* \, ([\circ]_* \, (\mathsf{q}[\langle u[\gamma]\rangle]))))] && =(\triangleright\beta_1) \\
& P[\gamma \circ (\mathsf{p} \circ \langle u[\gamma]\rangle), \mathsf{ass}_* \, ([\circ]_* \, ([\circ]_* \, (\mathsf{q}[\langle u[\gamma]\rangle])))] && =(\mathsf{ass}) \\
& P[(\gamma \circ \mathsf{p}) \circ \langle u[\gamma]\rangle, [\circ]_* \, ([\circ]_* \, (\mathsf{q}[\langle u[\gamma]\rangle]))] && =(-[-] \text{ and transport}) \\
& P[(\gamma \circ \mathsf{p}) \circ \langle u[\gamma]\rangle, [\circ]_* \, (([\circ]_* \, \mathsf{q})[\langle u[\gamma]\rangle])] && =(,\circ) \\
& P[(\gamma \circ \mathsf{p}, [\circ]_* \, \mathsf{q}) \circ \langle u[\gamma]\rangle] && \equiv \\
& P[(\gamma \circ \mathsf{p}, [\circ]_* \, \mathsf{q}) \circ \langle (\mathsf{Bool}[] \cdot \mathsf{Bool}[])_* \, (u[\gamma])\rangle] && =(\cdot_*) \\
& P[(\gamma \circ \mathsf{p}, [\circ]_* \, \mathsf{q}) \circ \langle \mathsf{Bool}[]_* \, (\mathsf{Bool}[]_* \, (u[\gamma]))\rangle] && =(\langle - \rangle \text{ and transport}) \\
& P[(\gamma \circ \mathsf{p}, [\circ]_* \, \mathsf{q}) \circ \mathsf{Bool}[]_* \, \langle \mathsf{Bool}[]_* \, (u[\gamma])\rangle] && =(- \circ - \text{ and transport}) \\
& P[(\mathsf{Bool}[]_* \, (\gamma \circ \mathsf{p}, [\circ]_* \, \mathsf{q})) \circ \langle \mathsf{Bool}[]_* \, (u[\gamma])\rangle] && \equiv
\end{aligned}
$$

Substitution-strict booleans in a strict CwF

$$
\begin{aligned}
&\mathsf{Bool} && : \mathsf{Ty}\,\Gamma \\
&\mathsf{Bool}[] && : \mathsf{Bool}[\gamma] \equiv \mathsf{Bool} \\
&\mathsf{true} && : \mathsf{Tm}\,\Gamma\,\mathsf{Bool} \\
&\mathsf{true}[] && : \mathsf{true}[\gamma] \equiv \mathsf{true} \\
&\mathsf{false} && : \mathsf{Tm}\,\Gamma\,\mathsf{Bool} \\
&\mathsf{false}[] && : \mathsf{false}[\gamma] \equiv \mathsf{false} \\
&\mathsf{ind} && : (P : \mathsf{Ty}\,(\Gamma \triangleright \mathsf{Bool})) \to \mathsf{Tm}\,\Gamma\,(P[\langle \mathsf{true} \rangle]) \to \mathsf{Tm}\,\Gamma\,(P[\langle \mathsf{false} \rangle]) \to \\
& && \ (b : \mathsf{Tm}\,\Gamma\,\mathsf{Bool}) \to \mathsf{Tm}\,\Gamma\,(P[\langle b \rangle]) \\
&\mathsf{ind}[] && : (\mathsf{ind}\,P\,p\,p'\,b)[\gamma] \equiv \mathsf{ind}\,(P[\gamma^{\uparrow}])\,(p[\gamma])\,(p'[\gamma])\,(b[\gamma]) \\
&\mathsf{Bool}\beta_1 && : \mathsf{ind}\,P\,p\,p'\,\mathsf{true} = p \\
&\mathsf{Bool}\beta_2 && : \mathsf{ind}\,P\,p\,p'\,\mathsf{false} = p'
\end{aligned}
$$