Generalizations of Hedberg's Theorem

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Overview

- Views on Type Theory
- Reminder: Equality in Type Theory
- Hedberg's Theorem
- Generalizations

Views on Type Theory

Formal system $(\Sigma, \Pi, \ldots)$

Programming

- computation
- verified programs

Proving

- foundation for constructive mathematics
- formalization
- e.g. a lot of homotopy theory has been formalized in HoTT

Reminder: Equality

Definitional Equality

Decidable equality for typechecking & computation; e. g.

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Propositional Equality

Equality needing a proof, e. g.

$$\forall\, m\, n\, .\, (m + n) = (n + m)$$

Reminder: Identity Types

Propositional equality

...is just an inductive type

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Formation

$$\frac{a, b \ : \ A}{a =_A b \ : \ \mathit{type}}$$

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$$\frac{a, b \;:\; A}{a =_A b \;:\; type}$$

Introduction

$$\frac{a \;:\; A}{refl_a \;:\; a =_A a}$$

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Propositional equality

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$$\frac{a, b : A}{a =_A b : type}$$

Introduction

$$\frac{a : A}{refl_a : a =_A a}$$

Elimination (J - Paulin-Mohring) for any $a : A$

$$\frac{P : (b : A) \to a =_A b \to \mathcal{U} \qquad m : P(a, refl_a)}{J_{(m,b,q)} : P(b, q)}$$

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Computation (β)

$$J_{(m,a,refl_a)} \equiv_\beta m$$

Uniqueness of Identity Proofs (UIP)

Given $a : A$.

- Can we show
  $(b, c : A) \to (p : a = b) \to (q : a = c) \to (b, p) = (c, q) \quad ?$

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  $(b, c : A) \rightarrow (p : a = b) \rightarrow (q : a = c) \rightarrow (b, p) = (c, q) \quad ?$

  Yes! Induction/J/"pattern matching" on (b, p) and (c, q)
  $\Rightarrow \qquad (a, refl_a) = (a, refl_a)$.

- Can we show $(p, q : a = a) \rightarrow p = q \quad ?$

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- Can we show $(p, q : a = a) \to p = q \quad ?$

  No!

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Axiom UIP (or K)

$$\frac{p,\ q\ :\ a = b}{\mathsf{UIP}\ :\ p = q}$$

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Advantages

- simple
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Disadvantages

- impossible to use the rich equality structure (as Homotopy Type Theory does to formalize axiomatic homotopy theory)
- incompatible with univalence (which allows us to identify isomorphic types)

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First, a definition:

Decidable Equality

$$\text{DecidableEquality}_A \;:\equiv\; \forall\, a\, b\,.\; (a = b \;+\; \neg\, a = b)$$

- Examples: $\mathbb{N}$, List_A if A has decidable equality
- Counterexamples: Colists (over a nonempty type), universes

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constant function

$$\mathrm{const}(f) \;:\equiv\; \forall\, a\, b\,.\, f(a) = f(b)$$

Hedberg's Theorem

$\mathsf{DecidableEquality}_A$

$\Downarrow$

there is a family $g_{ab} \; : \; a = b \to a = b$ of **constant** endofunctions

$\Updownarrow$

UIP_A

Strengthening Hedberg's Theorem

DecidableEquality is a very strong property.
How about something weaker? For example:

Separated

$$\forall\, a\, b\, .\, \neg\neg(a = b) \rightarrow a = b$$

With function extensionality,

$$\text{separated}_A \rightarrow \text{UIP}_A$$

Strengthening Hedberg's Theorem

We can still do better if we have *truncation*, aka *squash types* or *bracket types* (Awodey / Bauer).

Think of $\|A\|$ as the "squashed" version of A where we cannot distinguish the different inhabitants any more (similar to $\neg\neg A$).

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$$\forall\, a\, b\,.\ \|a = b\| \;\rightarrow\; a = b$$

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Generalizations

h-separated$_A$, i. e.
$\|a = b\| \to a = b$

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$\|X\| \to X$

?

there is a **constant**
$g : X \to X$

$\Uparrow$

all inhabitants of X are equal,
i. e. $(a, b : X) \to a = b$

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Applications

- We can define a new notion of *anonymous existence* that behaves similar to truncation $\|\cdot\|$, but is definable in Type Theory.
- We can show related theorems, such as:

If we have $\|X\| \to X$ for all types,
then all equalities are decidable.

Questions?

Thank you!