Internal parametricity, without an interval

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Identity type in type theory

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In general:

$$\frac{A : \mathsf{Type}}{\mathsf{Id}_A : A \to A \to \mathsf{Type}} \qquad \frac{a : A}{\mathsf{refl}_a : \mathsf{Id}_A\,a\,a} \qquad \dots$$

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$$
\begin{aligned}
\mathrm{Id}_{\mathbb{N}}\,0 \quad & 0 && := \top \\
\mathrm{Id}_{\mathbb{N}}\,(\mathrm{suc}\,m)\, & (\mathrm{suc}\,n) && := \mathrm{Id}_{\mathbb{N}}\,m\,n \\
\mathrm{Id}_{\mathbb{N}}\,0 \quad & (\mathrm{suc}\,n) && := \bot \\
\mathrm{Id}_{\mathbb{N}}\,(\mathrm{suc}\,n)\, & 0 && := \bot
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$$\mathsf{Id}_{A\times B}\,(a_0, b_0)\,(a_1, b_1) := \mathsf{Id}_A\, a_0\, a_1 \times \mathsf{Id}_B\, b_0\, b_1$$

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$$\mathrm{Id}_{A\to B}\, f_0\, f_1 :\equiv \forall a_0\, a_1\,.\, \mathrm{Id}_A\, a_0\, a_1 \to \mathrm{Id}_B\,(f_0\, a_0)\,(f_1\, a_0)$$

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$$\mathrm{Id}_{\mathsf{Type}}\,A\,B := (A \simeq B)$$
  Promises:
  - explainable: no interval, only low dimensional operations
  - computational univalence (unlike cubical type theory)
  - simple extension of Martin-Löf's type theory

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  - explainability, computation, simple extension

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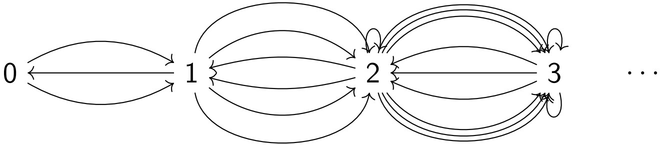
The category of BCH cubes:



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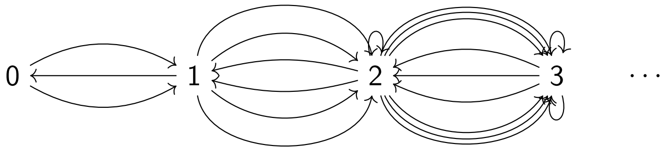

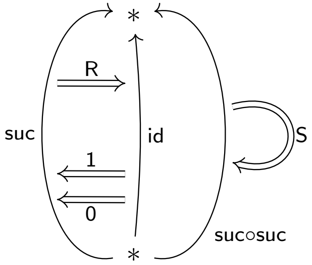

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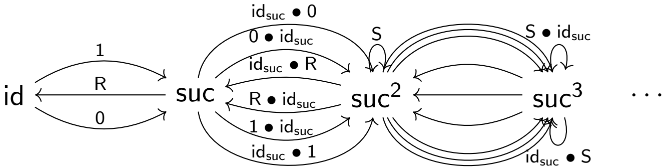
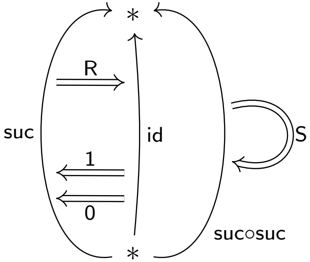
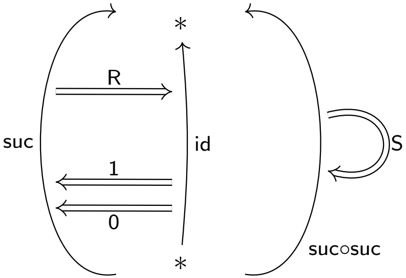




Syntax from semantics

The cube category $\square$:



Syntax from semantics

Structure on presheaves over $\square$:

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\mathrm{PSh}(\square) \arrow[d, bend right=70, "\mathrm{suc}^*"'] \arrow[d, "\mathrm{id}"] \arrow[d, bend left=70, "\mathrm{suc}^* \circ \mathrm{suc}^*"] \arrow[d, phantom, bend right=35, "\overset{R}{\Leftarrow} \quad \overset{1}{\Rightarrow} \quad \underset{0}{\Rightarrow}"] \arrow[d, phantom, bend left=100, "\overset{S}{\circlearrowleft}"] \\
\mathrm{PSh}(\square)
\end{tikzcd}
$$

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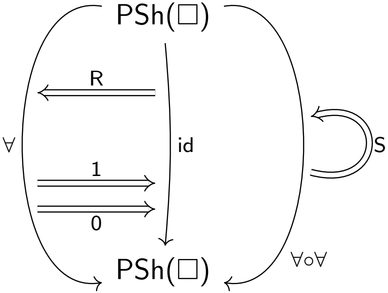

Syntax from semantics

Our global theory:

Structure on presheaves over $\square$:

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\mathrm{PSh}(\square) \arrow[d, bend right=80, "\forall"'] \arrow[d, "\mathrm{id}"] \arrow[d, bend left=80, "\forall\circ\forall"] \arrow[d, phantom, shift right=9ex, "\overset{\mathrm{R}}{\Leftarrow}\;\; \overset{1}{\Rightarrow}\;\; \underset{0}{\Rightarrow}"] \arrow[d, phantom, shift left=11ex, "\circlearrowleft \mathrm{S}"] \\
\mathrm{PSh}(\square)
\end{tikzcd}
$$

$$\frac{\vdash \Gamma}{\vdash \forall\Gamma} \qquad \frac{\sigma : \Delta \Rightarrow \Gamma}{\forall\sigma : \forall\Delta \Rightarrow \forall\Gamma}$$

$$\frac{\Gamma \vdash A}{\forall\Gamma \vdash \forall A} \qquad \frac{\Gamma \vdash t : A}{\forall\Gamma \vdash \forall t : \forall A}$$

$$\frac{\vdash \Gamma}{\begin{array}{c} \mathrm{R}_\Gamma : \Gamma \Rightarrow \forall\Gamma \\ 0_\Gamma : \forall\Gamma \Rightarrow \Gamma \\ 1_\Gamma : \forall\Gamma \Rightarrow \Gamma \\ \mathrm{S}_\Gamma : \forall\forall\Gamma \Rightarrow \forall\forall\Gamma \end{array}}$$

Syntax from semantics

Our local theory:

Structure on the standard model internal to $\mathrm{PSh}(\mathrm{PSh}(\square))$:

$$\begin{tikzcd}[row sep=huge]
\mathrm{Standard} \arrow[d, bend right=80, "\forall"'{name=A}] \arrow[d, "\mathrm{id}"{name=I}] \arrow[d, bend left=80, "\forall\circ\forall"{name=F}] \arrow[d, phantom, shift right=8ex, "\xLeftarrow{\mathrm{R}}"] \arrow[d, phantom, shift right=8ex, yshift=-3ex, "\xRightarrow{1}\ \underset{0}{\Rightarrow}"] \arrow[d, phantom, shift left=14ex, "\circlearrowleft \mathrm{S}"] \\
\mathrm{Standard}
\end{tikzcd}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall A} \qquad \frac{\Gamma \vdash f : A \to B}{\Gamma \vdash \mathrm{ap}\, f : \forall A \to \forall B}$$

$$\frac{\Gamma, x : A \vdash B \qquad a_2 : \forall A}{\Gamma \vdash \forall\!\!\mathrm{d}(x.B)\, a_2} \qquad \frac{\Gamma \vdash t : \Pi(x : A).B}{\Gamma \vdash \mathrm{apd}\, t : \Pi(a_2 : \forall A).\forall\!\!\mathrm{d}(x.B)\, a_2}$$

$$\frac{\Gamma \vdash A}{\begin{array}{c}\Gamma \vdash \mathrm{R_A} : A \to \forall A \\ \Gamma \vdash 0_\mathrm{A} : \forall A \to A \\ \Gamma \vdash 1_\mathrm{A} : \forall A \to A \\ \Gamma \vdash \mathrm{S_A} : \forall\forall A \to \forall\forall A\end{array}}$$

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- First structural type theory for BCH-cubes.
- Geometry is emergent, rather than built-in.
- We proved canonicity: every closed boolean is convertible to true or false.
- Ongoing and future work:
  - Prove normalisation
  - Replace spans by relations (Reedy fibrancy)
  - Add Kan operations = transport rule = symmetry, transitivity of Id
  - Implementation